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On small theories with a special type

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A type $p \in S(T)$ is called special, if there are $\bar{a}, \bar{b} \models p$ such that $\mathrm{tp}(\bar{a}/\bar{b})$ is isolated and non-algebraic, and $\mathrm{tp}(\bar{b}/\bar{a})$ is non-algebraic. In this paper, we will explain the result that any Ehrenfeucht theory has a special type. This result is due to Pillay in [1]. On the other hand, there are ω-stable examples with a special type[2, 3]. Here we will give another example with a special type. This is based on Sudoplatov's example.

Notation 0.1 $M, N, \ldots$ will denote L-structures and $A, B, \ldots$ subsets of structures. Elements of structures are denoted by $a, b, \ldots$ and finite tuples of elements are denoted by $\bar{a}, \bar{b}, \ldots$. If members of the tuple $\bar{a}$ come from A we sometimes write $\bar{a} \in A$. $A \subset_\omega B$ means that A is a finite subset of B. AB means $A \cup B$. $L(A)$ denotes the set of all formulas over A and L means $L(\emptyset)$. $S(A)$ denotes the set of all types over A and $S(T)$ means $S(\emptyset)$. The set of all algebraic elements over A in M is denoted by $\mathrm{acl}_M(A)$.

1 Proposition

In what follows, T is a complete theory in a coutable language L.

Definition 1.1 Let $p \in S(T)$ be nonisolated. Then p is said to be special, if there are $\bar{a}, \bar{b} \models p$ such that

- $\mathrm{tp}(\bar{b}/\bar{a})$ is isolated and non-algebraic;
- $\mathrm{tp}(\bar{a}/\bar{b})$ is non-isolated.

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Example 1.2 The following example is well-known and has a special type: Let

$$T = \mathrm{Th}(\mathcal{Q}, <, 0, 1, 2, ...)$$

and let $\mathcal{M}$ be a big model. Let $p = \{n < x\}_{n \in \omega}$ and take realizations $a, b \models p$ with $a < b$. Then $\mathrm{tp}(a/b)$ is nonisolated, and $\mathrm{tp}(b/a)$ is isolated and nonalgebraic. Hence p is special.

The example stated above is an Ehrenfeucht theory (see Definition 1.13). In this section, we want to show that any Ehrenfeucht theory has a special type (Proposition 1.14). To prove the result, we need some preparation.

Definition 1.3

1. The Cantor-Bendixson rank $\mathrm{CB}(\varphi)$ of a formula $\varphi(\bar{x}) \in L$ is defined as follows:
   - If $\varphi(\bar{x})$ is consistent, then $\mathrm{CB}(\varphi) \geq 0$;
   - Let β be limit. Then $\mathrm{CB}(\varphi) \geq \beta$, if $\mathrm{CB}(\varphi) \geq \alpha$ for any $\alpha < \beta$;
   - $\mathrm{CB}(\varphi) \geq \alpha + 1$ if there are formulas $\varphi_i(\bar{x}) \in L$ $(i \in \omega)$ such that
     (a) $\models \neg \exists \bar{x}(\varphi_i(\bar{x}) \wedge \varphi_j(\bar{x}))$ for each $i, j \in \omega$ with $i \neq j$;
     (b) $\mathrm{CB}(\varphi \wedge \varphi_i) \geq \alpha$ for each $i \in \omega$.
   - If $\mathrm{CB}(\varphi) \geq \alpha$ for all α, then we say $\mathrm{CB}(\varphi) = \infty$;
   - If $\mathrm{CB}(\varphi) \geq \alpha$ and $\mathrm{CB}(\varphi) \ngeq \alpha + 1$, then we say $\mathrm{CB}(\varphi) = \alpha$.
2. The rank $\mathrm{CB}(p)$ of a type $p \in S(T)$ is defined to be $\min\{\mathrm{CB}(\varphi) : \varphi \in p\}$.
3. The degree $\deg(\varphi)$ of φ is defined to be the greatest $m \in \omega$ such that there are distinct $p_1, ..., p_m \in S(T)$ with $\mathrm{CB}(p_i) = \mathrm{CB}(\varphi)$ for $i = 1, ..., m$.
4. Let $\mathrm{CB}(\bar{a})$ denote $\mathrm{CB}(\mathrm{tp}(\bar{a}))$.

Note 1.4 If $\bar{a} \in \mathrm{acl}(\bar{b})$, then $\mathrm{CB}(\bar{b}) = \mathrm{CB}(\bar{a}\bar{b})$.

Definition 1.5 A theory T is said to be small, if $S(T)$ is countable.

Note 1.6 If T is small, then each formula $\varphi(\bar{x}) \in L$ has the CB-rank.

The following lemma was suggested by Anand Pillay, and it can be found in [1].

Lemma 1.7 Suppose that T is small. Let $p \in S(T)$ and $\bar{a}, \bar{b} \models p$. If $\mathrm{tp}(\bar{b}/\bar{a})$ is algebraic, then $\mathrm{tp}(\bar{a}/\bar{b})$ is isolated.

Proof. Assume that T is small. By Note 1.6, we can take a formula $\varphi(\bar{x}, \bar{y}) \in \mathrm{tp}(\bar{a}\bar{b})$ with

$$\mathrm{CB}(\bar{a}\bar{b}) = \mathrm{CB}(\varphi(\bar{x}, \bar{y})) \text{ and } \deg(\varphi(\bar{x}, \bar{y})) = 1.$$

Since $\mathrm{tp}(\bar{b}/\bar{a})$ is algebraic, we can assume that

$$\models \varphi(\bar{a}', \bar{b}') \text{ implies } \bar{b}' \in \mathrm{acl}(\bar{a}').$$

We want to show that

$$\varphi(\bar{x}, \bar{b}) \vdash \mathrm{tp}(\bar{a}/\bar{b}).$$

Take any $\bar{a}' \models \varphi(\bar{x}, \bar{b})$. Clearly we have

$$\mathrm{CB}(\bar{a}'\bar{b}) \leq \mathrm{CB}(\bar{a}\bar{b}).$$

Since $\bar{b} \in \mathrm{acl}(\bar{a}')$, by Note 1.4, we have

$$\mathrm{CB}(\bar{b}) \leq \mathrm{CB}(\bar{a}').$$

Then we have

$$\begin{aligned} \mathrm{CB}(\bar{b}) &\leq \mathrm{CB}(\bar{a}') \\ &\leq \mathrm{CB}(\bar{a}'\bar{b}) \\ &\leq \mathrm{CB}(\bar{a}\bar{b}) \\ &\leq \mathrm{CB}(\bar{a}) \quad (\text{since } \bar{b} \in \mathrm{acl}(\bar{a})) \\ &= \mathrm{CB}(\bar{b}) \quad (\text{since } \mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})). \end{aligned}$$

Hence

$$\mathrm{CB}(\bar{a}'\bar{b}) = \mathrm{CB}(\bar{a}\bar{b}).$$

Since $\deg(\varphi(\bar{x}, \bar{y})) = 1$, we have

$$\mathrm{tp}(\bar{a}'\bar{b}) = \mathrm{tp}(\bar{a}\bar{b}).$$

Therefore we have

$$\bar{a}' \models \mathrm{tp}(\bar{a}/\bar{b}).$$

Definition 1.8 Let $p \in S(T)$ be non-isolated. Then p is said to be powerful, if any model realizing p realizes every type over $\emptyset$.

Note 1.9 It is known that any Ehrenfeucht theory has a poweful type.

Definition 1.10 $\mathrm{tp}(b/a)$ is said to be semi-isolated, if there is a formula $\varphi(x,a) \in \mathrm{tp}(b/a)$ with $\varphi(x,a) \vdash \mathrm{tp}(b)$.

Note 1.11 It is clear that

- every isolated type is semi-isolated;
- if $\mathrm{tp}(a/b)$ and $\mathrm{tp}(b/c)$ are semi-isolated, then $\mathrm{tp}(a/c)$ is semi-isolated. (Transitivity)

The following lemma is known, however, for completeness, we give a proof.

Lemma 1.12 Any non-isolated type $p \in S(T)$ has realizations $\bar{b}, \bar{b}'$ such that $\mathrm{tp}(\bar{b}'/\bar{b})$ is not semi-isolated.

Proof. Take any $\bar{b} \models p$, and let

$$\Phi(\bar{x}) = \{\neg\varphi(\bar{x},\bar{b}) \in L(\bar{b}) : \varphi(\bar{x},\bar{b}) \vdash p(\bar{x})\}.$$

First, we want to show that

$$p(\bar{x}) \cup \Phi(\bar{x}) \text{ is consistent.}$$

If not, then there are $\neg\varphi_1, ..., \neg\varphi_n \in \Phi$ with

$$p \vdash \varphi_1 \vee ... \vee \varphi_n.$$

By compactness, there is a $\psi \in p$ with

$$\psi \vdash \varphi_1 \vee ... \vee \varphi_n.$$

Since $\varphi_1 \vee ... \vee \varphi_n \vdash p$, we have $\psi \vdash p$. A contradiction. So we can take a realization

$$\bar{b}' \models p(\bar{x}) \cup \Phi(\bar{x}).$$

Then $\mathrm{tp}(\bar{b}'/\bar{b})$ is not semi-isolated.

Definition 1.13 A theory T is said to be Ehrenfeucht, if it has finitely many countable models, and is not ω-categorical. Note that every Ehrenfeucht theory is small.

The following proposition can be obtained by Lemma 1.7, and it was also suggested by Anand Pillay.

Proposition 1.14 Any Ehrenfeucht theory has a special type.

Proof. Assume that T is Ehrenfeucht. By note 1.9, there is a powerful type $p(\bar{x})$. By Lemma 1.12, we can take $\bar{b}, \bar{b}' \models p$ such that

$$\mathrm{tp}(\bar{b}'/\bar{b}) \text{ is not semi-isolated.}$$

Since p is powerful, we can take $\bar{a} \models p$ such that

$$\mathrm{tp}(\bar{b}\bar{b}'/\bar{a}) \text{ is isolated.}$$

By the transitivity of semi-isolation,

$$\mathrm{tp}(\bar{a}/\bar{b}) \text{ is nonisolated.}$$

By Lemma 1.7, $\mathrm{tp}(\bar{b}/\bar{a})$ is not algebraic. Hence p is special.

2 Example

Proposition 1.14 says that any Ehrenfeucht theory has a special type. In fact, Example 1.2 is Ehrenfeucht and then has a special type. However, this example is unstable. So the following question arise naturally:

Question 2.1 Is there a (small) stable theory with a special type?

For this question, Anand Pillay suggested that he had had an ω-stable example with special type [2]. Also, Sergey Sudoplatov told me that he had also obtained an example satisfying the same condition [3]. In this section, we will give an ω-stable theory with a special type. This example is based on Sudoplatov's one, but it is constructed by the Hrushovski amalgamation construction.

Here, by a digraph (or directed graph) we mean a graph (A, R^A) satisfying

- $A \models \forall x \forall y (R(x,y) \to \neg R(y,x))$;

- $A \models \forall x \forall y (R(x,y) \to x \neq y)$,

where $R^A = \{ab \in A : A \models R(a,b)\}$, Let $Q(x,y)$ denote $R(x,y) \vee R(y,x)$.

Let $L = \{R(*,*), U_0(*), U_1(*), ...\}$, and $\mathbf{K}$ a class of all finite L-structures A with the following property:

1. (A, R^A) is a digraph;
2. (A, R^A) has no cycles, i.e., there is no sequence $a_0 a_1 ... a_n$ in A with $A \models Q(a_0, a_1) \wedge Q(a_1, a_2) \wedge ... \wedge Q(a_n, a_1)$ for each $n \in \omega$;
3. $U_0^A \subset U_1^A \subset \cdots$;
4. For any $i \in \omega$, if $A \models R(a,b) \wedge U_i(b)$ then there is some $j \leq i$ with $A \models U_j(a)$.

For $A \in \mathbf{K}$, a predimension of A is defined by

$$\delta(A) = |A| - \alpha |R^A|,$$

where $\alpha \in (0,1]$. In our setting, let $\alpha = 1$. Let $\delta(B/A)$ denote $\delta(B \cup A) - \delta(A)$. For $A \subset B \in \mathbf{K}$, A is said to be strong (or closed) in B (write $A \leq B$), if

$$\delta(X/A) \geq 0 \text{ for any } X \subset B.$$

For A, B, C with $A = B \cap C$, $B \perp_A C$ means

$$R^{B \cup C} = R^B \cup R^C.$$

When $B \perp_A C$, a graph $B \cup C$ is denoted by $B \oplus_A C$.

Note 2.2 If $A \leq B \in \mathbf{K}$ and $b \in B - A$ is connected with A, then there is a unique $a \in A$ such that $bb_1...b_n a$ is a path between a and b, i.e., $B \models Q(b, b_1) \wedge Q(b_1, b_2) \wedge ... \wedge Q(b_n, a)$ for some distinct $b_1, b_2, ..., b_n \in B - A$.

Proof. Suppose that there would be another path $bb'_1 b'_2 ... b'_m a'$ for some $a' \in A$ and $b'_1, b'_2, ..., b'_m \in B - A$. Then we have

$$\delta(bb_1...b_n b'_1 ... b'_m / aa') = -1 < 0,$$

and hence $A \not\leq B$. A contradiction.

Lemma 2.3 If $A \leq B \in \mathbf{K}$, $A \subset C \in \mathbf{K}$ and $B \perp_A C$, then $D = B \oplus_A C \in \mathbf{K}$.

Proof. Take any $A, B, C \in \mathbf{K}$ with

$$A \leq B, A \subset C \text{ and } B \perp_A C.$$

Let $D = B \oplus_A C$. Clearly D satisfies conditions 1,3 and 4 of the definition of $\mathbf{K}$. Suppose that D would have a cycle S. Since B and C have no cycles, there are $b \in S \cap (B - A)$ and distinct $a, a' \in S \cap A$ such that

$$b \text{ is connected with both of } a \text{ and } a'.$$

By Note 2.2, we have $A \not\leq B$. A contradiction. Hence $D \in \mathbf{K}$.

Let $\overline{\mathbf{K}}$ be a class of (possibly infinite) L-structures M satisfying $F \in \mathbf{K}$ for any $F \subset_\omega M$. Let $A \subset B \in \overline{\mathbf{K}}$, we define $A \leq B$, if

$$A \cap F \leq B \cap F \text{ for any } F \subset_\omega B.$$

The closure $\mathrm{cl}_B(A)$ of A in B is defined by

$$\mathrm{cl}_B(A) = \bigcap \{C \subset B : A \subset C \leq B\}.$$

Note 2.4 For any finite $A \subset M \in \overline{\mathbf{K}}$, $\mathrm{cl}_M(A)$ is finite, because α is 1 (or rational).

Definition 2.5 A countable L-structure M is said to be $(\mathbf{K}, \leq)$-generic, if

1. $M \in \overline{\mathbf{K}}$;
2. if $A \leq B \in \mathbf{K}$ and $A \leq M$ then there is a $B' \cong_A B$ with $B' \leq M$;
3. if $A \subset_\omega M$ then $\mathrm{cl}_M(A)$ is finite.

By Lemma 2.3, $(\mathbf{K}, \leq)$ has the (free) amalgamation property, i.e., if $A \leq B \in \mathbf{K}$ and $A \leq C \in \mathbf{K}$ then $B \oplus_A C \in \mathbf{K}$. Then it can be seen that there is the $(\mathbf{K}, \leq)$-generic structure M.

In what follows, M is the generic structure for $(\mathbf{K}, \leq)$, $T = \mathrm{Th}(M)$, and $\mathcal{M}$ is a big model of T.

For $n \in \omega$ and $A \subset B$ we define $A \leq_n B$ by $A \leq X \cup A$ for any $X \subset B - A$ with $|X| \leq n$. Also, for A, A', we define $A \cong_n A'$ by A and A' are isomorphic in the language $\{R, U_0, ..., U_n\}$.

Note 2.6 If $A \leq B \in \mathbf{K}$ and $A \leq \mathcal{M}$, then there is a $B' \cong_A B$ with $B' \leq \mathcal{M}$.

Proof. For $n \in \omega$ and $C \subset_\omega \mathcal{M}$, let $\theta^n_C(X)$ be a formula expressing that

$$X \cong_n C \text{ and } X \leq_n \mathcal{M}.$$

Take any $A, B \in \mathbf{K}$ with $A \leq B$ and $A \leq \mathcal{M}$. First, we want to show that

$$M \models \forall X(\theta^n_A(X) \to \exists Y \theta^n_{AB}(XY))$$

for each $n \in \omega$. Take any A' with $M \models \theta^n_A(A')$. Let $C' = \mathrm{cl}_M(A')$. Note that C' is finite and $A' \leq_n C'$. It is easily checked that there is a $B^* \in \mathbf{K}$ with $B^*A' \cong_n BA$. Then we have

$$C' \leq B^* \oplus_{A'} C' \in \mathbf{K}.$$

By genericity of M, we can assume that $B^*C' \leq M$, and then $M \models \theta^n_{AB}(A'B^*)$. Hence we have

$$\mathcal{M} \models \forall X(\theta^n_A(X) \to \exists Y \theta^n_{AB}(XY))$$

From this it follows that

$$\{\theta^n_{AB}(AY)\}_{n\in\omega} \text{ is consistent.}$$

So we can take its realization B'. Then B' is as required.

Lemma 2.7 M is saturated.

Proof. Take any $A \subset_\omega M$ and any type $p \in S(A)$. We want to show that

$$p \text{ is realized by } M.$$

Without loss of generality, we can assume $A \leq M$, and moreover $A = \emptyset$. Take a realization $\bar{b} \models p$ in $\mathcal{M}$. By Note 2.4, $B_0 = \mathrm{cl}(\bar{b})$ is finite. By genericty of M, we can take B_0' with

$$B_0' \leq M \text{ and } B_0' \cong B_0.$$

Take any $c' \in M - B_0'$ and let $B_1' = \mathrm{cl}_M(c'B_0')$. Let B_1 be such that $B_1B_0 \cong B_1'B_0'$.

Note that $B \leq B_1 \in \mathbf{K}$. By Note 2.6, there is a B_1^* with

$$B_1^* \leq \mathcal{M} \text{ and } B_0'B_1^* \cong B_0B_1.$$

Iterationg this process, for each $i \in \omega$ there is an isomorphisim $\sigma_i : B_i \to B_i'$ such that

- $B_0 \leq B_1 \leq B_2 \leq ... \leq \mathcal{M}$;
- $B_0' \leq B_1' \leq B_2' \leq ... \leq M$;
- $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset ....$

Therefore we have

$$\mathrm{tp}(B_0) = \mathrm{tp}(B_0').$$

Take $\bar{b}'$ with $\mathrm{tp}(B_0\bar{b}) = \mathrm{tp}(B_0'\bar{b}')$. Hence p is realized by $\bar{b}' \in M$.

Note 2.8 Let $A, B \leq \mathcal{M}$ and $A \cong B$. Then, by saturation of M and the back and forth argument, we have $\mathrm{tp}(A) = \mathrm{tp}(B)$.

Definition 2.9 For $\bar{a}, \bar{b} \in \mathcal{M}$, a dimension of $\bar{a}$ is defined by $d(\bar{a}) = \delta(\mathrm{cl}(\bar{a}))$, and $d(\bar{a}\bar{b}) - d(\bar{b})$ is denoted by $d(\bar{a}/\bar{b})$. For an infinite $B \subset \mathcal{M}$, $d(\bar{a}/B)$ is defined by $d(\bar{a}/B) = \min\{d(\bar{a}/\bar{b}) : \bar{b} \in B\}$.

Note 2.10 Let $\bar{b} \in \mathcal{M}$ and $A, C \subset \mathcal{M}$ with $A = \mathrm{cl}(\bar{b}A) \cap C$ and $A \leq C \leq \mathcal{M}$. Then it can be seen that the following are equivalent:

1. $d(\bar{b}/C) = d(\bar{b}/A)$;
2. $\mathrm{cl}(\bar{b}A) \cup C \leq \mathcal{M}$ and $\mathrm{cl}(\bar{b}A) \perp_A C$.

Lemma 2.11 T is ω-stable.

Proof. Since M is saturated, it is enough to show that

$$S(M) \text{ is countable.}$$

Take any $p \in S(M)$ and $\bar{e} \models p$ in $\mathcal{M}$. Then there is a finite $A \leq M$ with

$$d(\bar{e}/M) = d(\bar{e}/A) \text{ and } \mathrm{cl}(\bar{e}A) \cap M = A.$$

Take any $\bar{e}' \models \mathrm{tp}(\bar{e}/A)$ with

$$d(\bar{e}'/M) = d(\bar{e}'/A) \text{ and } \mathrm{cl}(\bar{e}'A) \cap M = A.$$

Then it is clear that

$$\mathrm{cl}(\bar{e}A) \cong_A \mathrm{cl}(\bar{e}'A).$$

By Note 2.10, we have

$$\mathrm{cl}(\bar{e}A) \perp_A M \text{ and } \mathrm{cl}(\bar{e}'A) \perp_A M.$$

Therefore we have

$$\mathrm{cl}(\bar{e}A) \cong_M \mathrm{cl}(\bar{e}'A).$$

Again, by Note 2.10, we have

$$\mathrm{cl}(\bar{e}A)M, \mathrm{cl}(\bar{e}'A)M \leq \mathcal{M}.$$

By Note 2.8, we have

$$\mathrm{tp}(\bar{e}/M) = \mathrm{tp}(\bar{e}'/M).$$

This means that any type over M is determined by a type over A for some finite $A \subset M$. By Lemma 2.7, T is small, and then $S(A)$ is countable for each finite A. Therefore

$$|S(M)| \leq |\{A : A \subset_\omega M\}| \cdot \max\{|S(A)| : A \subset_\omega M\} = \aleph_0 \cdot \aleph_0 = \aleph_0.$$

Hence T is ω-stable.

Lemma 2.12 T has a special type.

Proof. Let

$$p(x) = \{\neg U_0(x), \neg U_1(x), ...\}.$$

Then p is complete, since any 1-element is closed in $\mathcal{M}$. Take $a, b \models p$ with $M \models R(a, b)$ and $ab \leq M$. First, we show that

$$\mathrm{tp}(b/a) \text{ is isolated and non-algebraic.}$$

In fact, we can see that $R(a, x)$ isolates $\mathrm{tp}(b/a)$. Take any b' with $\models R(a, b')$. Since $a \models p$, by condition 4 of the definition of **K**, we have $b' \models p$, and then

$$b'a \cong ba.$$

On the other hand, by condition 2 of the definition of **K**, we have $ab' \leq \mathcal{M}$. By Note 2.8, we have

$$\mathrm{tp}(b'/a) = \mathrm{tp}(b/a).$$

Hence $\mathrm{tp}(b/a)$ is isolated. On the other hand, by genericity of M, for each $n \in \omega$ there are $b_1, b_2, ..., b_n \in M$ with

$$R(a, b_i) \text{ and } ab_i \leq ab_1...b_n \leq \mathcal{M}$$

for any $i = 1, ..., n$. Hence $\mathrm{tp}(b/a)$ is non-algebraic. Next we show that

$$\mathrm{tp}(a/b) \text{ is non-isolated.}$$

It can be easily seen that

$$\{R(x, b)\} \cup p(x) \vdash \mathrm{tp}(a/b).$$

Suppose that $\mathrm{tp}(a/b)$ would be isolated. Then there is some $n \in \omega$ such that

$$R(x, b) \wedge \neg U_n(x) \vdash \mathrm{tp}(a/b).$$

On the other hand, by the definition of **K**, there is a' with

$$a'b \models R(a', b) \wedge U_{n+1}(a') \wedge \neg U_n(a') \text{ and } a'b \in \mathbf{K}.$$

Since $b \leq a'b$, we can assume that $a'b \leq \mathcal{M}$. Then we have

$$\models R(a', b) \wedge \neg U_n(a') \text{ and } \mathrm{tp}(a'/b) \neq \mathrm{tp}(a/b).$$

This is a contradiction. Hence $\mathrm{tp}(a/b)$ is non-isolated.

References

[1] A. Pillay, PhD thesis (1978)

[2] A. Pillay, unpbulished note

[3] S. V. Sudoplatov, Powerful types in small theories, Siberian Mathematical Journal (1990)